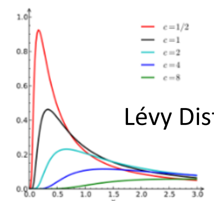
Scaling behaviour in the dynamics of an economic index

- **Rosario N. Mantegna & H. Eugene Stanley**

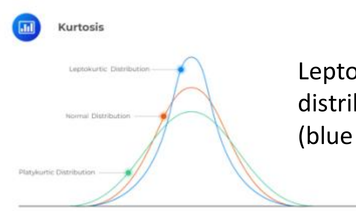
- **1995, Nature**
- **Link to paper**

Abstract Summary:
In this paper the authors show that the fluctuations of the S&P 500 index can be described by a non-gaussian process (in the center of the distribution) that corresponds to a Levy stable process (the continuous-time analog of a random walk).

Intro & Lit/Review:
- Most widely accepted models state that the variation of *share price* is a random process
- For the distribution of index returns (which is the difference between two successive logarithms of price), the prominent proposals include:
  - A normal distribution - Gaussian
  - A Lévy stable distribution - the continuous-time analog of a random walk
  - Leptokurtic distributions generated by a mixture of distributions
  - ARCH/GARCH models
    - Major difference between them lies within the tails of the distribution



Lévy Distribution

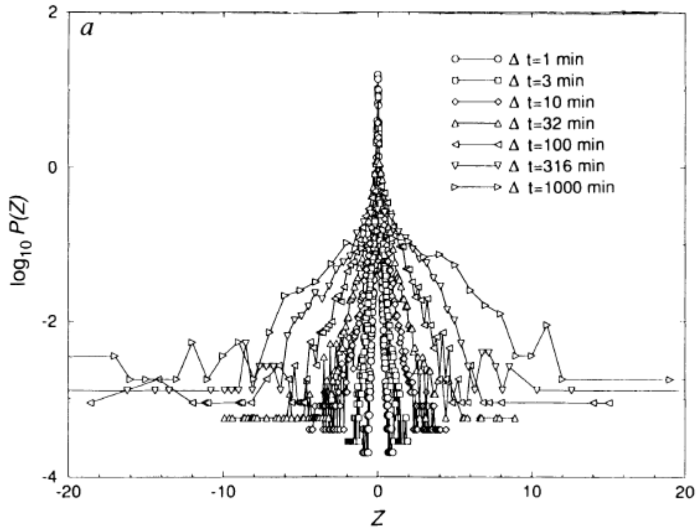


Leptokurtic distribution (blue line)

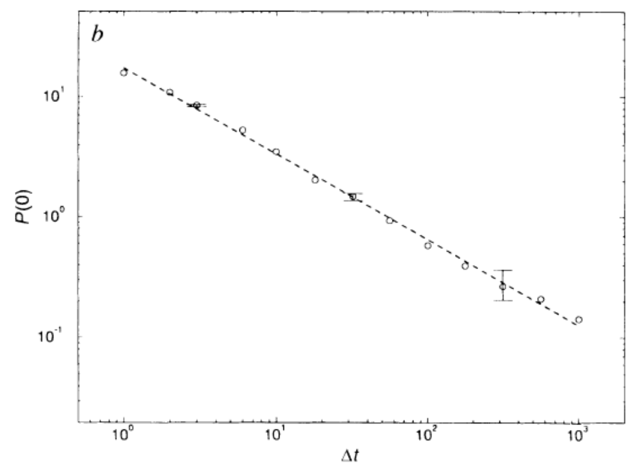
Methods:
- Study NYSE, S&P 500 index over 6-year period (Jan 1984-Dec. 1989)
- Find that the S&P 500 is a stochastic process remarkably well described by a Lévy stable symmetrical process (except for the rarest events)
  - Specifically, the dynamics of the central part of the distribution are predicted well by a Lévy stable process
- 1.4M records are studied
  - Time interval between recorded indices is **not consistent**
  - ~1 minute from '84-'85 and ~15 sec. from '86-'89
- Time is considered continuous from start of market open to close of market, from day to day
- Records are separated by a time interval: $\Delta t \pm \varepsilon \Delta t$ where ε is always less than .035
- Value of S&P 500 is denoted as $y(t)$
- Successive variations of S&P index is $Z(t) \equiv y(t) - y(t - \Delta t)$
  - **The distribution of successive variations is described as value at time *t* minus the value at the previous point *t*, giving us the *CHANGE of the S&P index over time.***
- After calculating the distribution $Z(t)$ for all time intervals, they select Δt values that are equally spaced between one another from'84 to '89.

- The Δt intervals were chosen at logarithmically larger and larger intervals between 1 min and 1000 mins (i.e. 1 min, 3 min, 10 mins, 32 mins, 100 mins, 316 mins, 1000 mins) and utilized to calculate probability distributions

- Plotting these probability distributions returns the figure to the right and we learn the following about the distributions:
  - Roughly symmetrical about zero
  - Spread wider as Δt increases
  - Leptokurtic (tails larger than expected for a normal process)
  - Yes, this figure is **atrocious**



- To study the distributions, they decide to study the "probability of return"
  - a.k.a. $P(Z) = 0$
- Plotting $P(0)$ versus Δt returns the figure to the right and we learn:
  - Data fits well by a straight line with a slope of -.712+/-.5
  - This finding agrees with the theoretical model of a Lévy walk or Lévy flight.
    - See below and [here] for more details



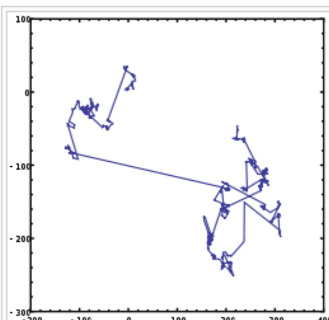


Figure 1. An example of 1000 steps of a Lévy flight in two dimensions. The origin of the motion is at [0,0], the angular direction is uniformly distributed and the step size is distributed according to a Lévy (i.e. stable) distribution with $\alpha = 1$ and $\beta = 0$ which is a Cauchy distribution. Note the presence of large jumps in location compared to the Brownian motion illustrated in Figure 2.

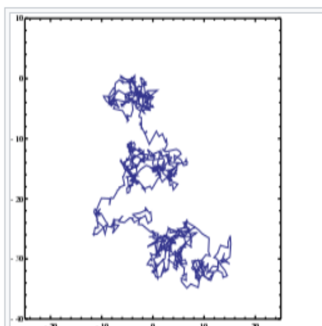


Figure 2. An example of 1000 steps of an approximation to a Brownian motion type of Lévy flight in two dimensions. The origin of the motion is at [0, 0], the angular direction is uniformly distributed and the step size is distributed according to a Lévy (i.e. stable) distribution with $\alpha = 2$ and $\beta = 0$ (*i.e.*, a normal distribution).

- Using the Lévy stable symmetrical distribution (below), they calculate the scaling factor
    - α = 1.40 +/- 0.05

$$L_\alpha(Z, \Delta t) \equiv \frac{1}{\pi} \int_0^\infty \exp(-\gamma \Delta t q^\alpha) \cos(qZ) dq \qquad (1)$$

of index α and scale factor γ at $\Delta t = 1$, where $\exp(-\gamma \Delta t |q|^\alpha)$ is the characteristic function of the symmetrical stable process, then the probability of return is given by

$$P(0) \equiv L_\alpha(0, \Delta t) = \frac{\Gamma(1/\alpha)}{\pi \alpha (\gamma \Delta t)^{1/\alpha}} \qquad (2)$$

where Γ is the Gamma function. By using the value -0.712 ± 0.025 from the data of Fig. 1*b* we obtain the index $\alpha = 1.40 \pm 0.05$.

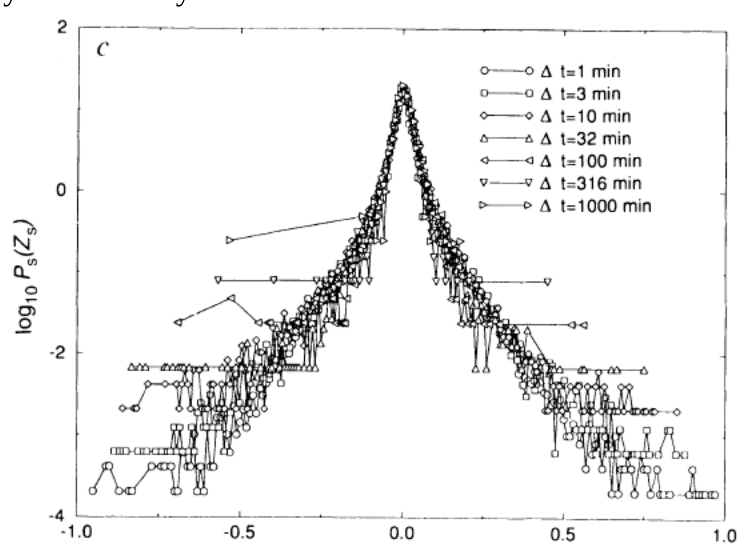
We also check if the scaling extends over the entire probability distribution as well as $Z = 0$. To this end, we first note that Lévy stable symmetrical distributions rescale under the transformations

$$Z_s \equiv \frac{Z}{(\Delta t)^{1/\alpha}} \qquad (3a)$$

and

$$L_\alpha(Z_s, 1) \equiv \frac{L_\alpha(Z, \Delta t)}{(\Delta t)^{-1/\alpha}} \qquad (3b)$$

- After the above scaling transformations they utilize the acquired scaling factor to then collapse (scale) all data down to the 1 min level to see how the different scales fit and find that they fit extremely well



- Based on the above figure they conclude, *"… that a Lévy distribution describes well the dynamics of the probability distribution* P(Z) *of the random process over time intervals spanning three orders of magnitude."*

- Finally, they investigate the possibility that the scaling parameter may fluctuate overtime and plot the figure below, which illustrates *P(0)* as a function of Δt, for each of data that was gathered.

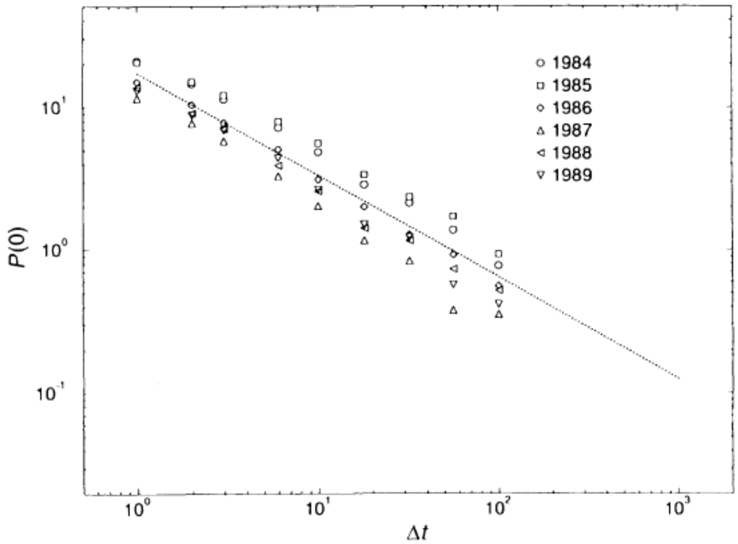


FIG. 3 Probability of return *P*(0) of the S&P 500 index variations as a function of the time sampling intervals observed in different years. The same scaling behaviour of the complete set of data (dotted line) is observed each year (data are parallel to the dotted line). The scale factor γ is slowly time-dependent, as the vertical positions of the probability of return change from year to year.

- It is highlighted that the scale factor γ varies somewhat from year to year and is higher (lower position on the graph) for periods with higher volatility.
- They additionally calculate this value at the monthly level and indicate that the results did not change much.