*Modeling urban growth patterns*

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- **1995, Nature**
- **Link to paper**

**General Takeaway:**
Diffusion-limited aggregation models have been utilized to represent the growth of cities. Typically, these models suggest that almost all of the growth will be seen at the tips of the clusters branches. However, this paper shows that an alternative model represents this growth process more effectively. Specifically, when units of development are *correlated* as opposed to being *added to the cluster at random.*

- The physical model they utilize corresponds to the correlated percolation model in the presence of a density gradient.

Model Takes into Account Two Major Aspects:

1. Data on the population density $\rho(r)$ of actual urban systems are known to conform to the relation $\rho(r) \equiv \rho_0 e^{-\lambda r}$, where:
   a. $r$ = the distance from the compact core
   b. $\lambda$ = the density gradient
   c. Thus, **development units are positioned with occupancy probability**…
      i. $\rho(r) = {}^{\rho(r)}/_{\rho_0}$
      ii. This behaves in the same fashion as seen in real world observations of cities
2. Developmental units in the real world are not placed at random. Rather, they are placed with some correlation to the location of a previous development
   a. i.e the probability of a development in location *x* is higher if the locations around *x* have been developed
   b. To quantify this idea, they utilize a correlated percolation model
   c. When correlations are so small as to be negligible, a site at position $\boldsymbol{r}$ is occupied if the occupancy variable $\boldsymbol{u}(\boldsymbol{r})$ – an uncorrelated random number – is smaller than the occupation probability $p(\boldsymbol{r})$
   d. They introduce correlations among the variables by convoluting the uncorrelated variables $u(\boldsymbol{r})$ with a suitable power-law kernel, and define a new set of random variables $\eta(\boldsymbol{r})$ with long-range power-law correlations that decay as $r^{-a}$, where $r \equiv |\boldsymbol{r}|$ ---- > (KEEP IN MIND THAT ONE *r* IS BOLD AND THE OTHER IS NOT)
      i. The basis for using a power-law has to do with the relation between distance from the occupied neighborhood.

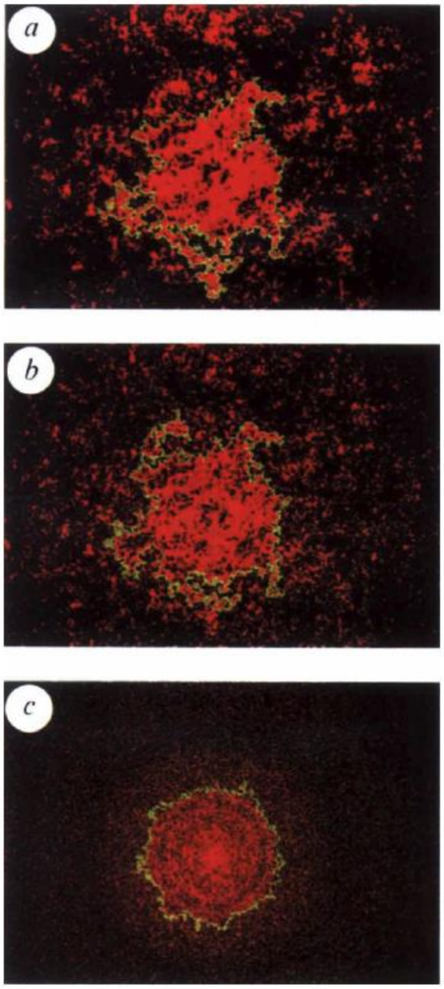



FIG. 1 *a–c*, Simulations of urban systems for different degrees of correlations. Red indicates the urban areas, and light green shows the external perimeter or urban boundary of the largest cluster connected to the CBD. In all the panels, we fix the value of the density gradient to be $\lambda = 0.009$. *a*, *b*, Two different examples of interactive systems of cities for correlation exponents $\alpha = 0.6$ and $\alpha = 1.4$, respectively. The development units are positioned with a probability that decays exponentially with the distance from the core. The units are located not randomly as in percolation, but rather in a correlated fashion depending on the neighbouring occupied areas. The correlations are parametrized by the exponent $\alpha$. The strongly correlated case corresponds to small $\alpha$ ($\alpha \to 0$). When $\alpha > d$, where $d$ is the spatial dimension of the substrate lattice ($d = 2$ in our case), we recover the uncorrelated case. Notice the tendency to more compact clusters as we increase the degree of correlations ($\alpha \to 0$). *c*, As a zeroth-order approximation, one might consider the morphology predicted in the extreme limit when development units are positioned at random, rather than in the correlated way shown in *a* and *b*. The results for this crude approximation of a non-interactive (uncorrelated) system of cities clearly display a drastically different morphology than found from data on real cities (such as shown in Fig. 2*a*). The non-interactive limit looks unrealistic in comparison with real cities, for the lack of interactions creates an urban area characterized by many small towns spread loosely around the core.

Basically, what the above shows us is how the growth patterns change depending on the power-law exponent. “a” shows the strongly correlated case, while “c” shows the other extreme.

One the next page, they model this against real world data over time…

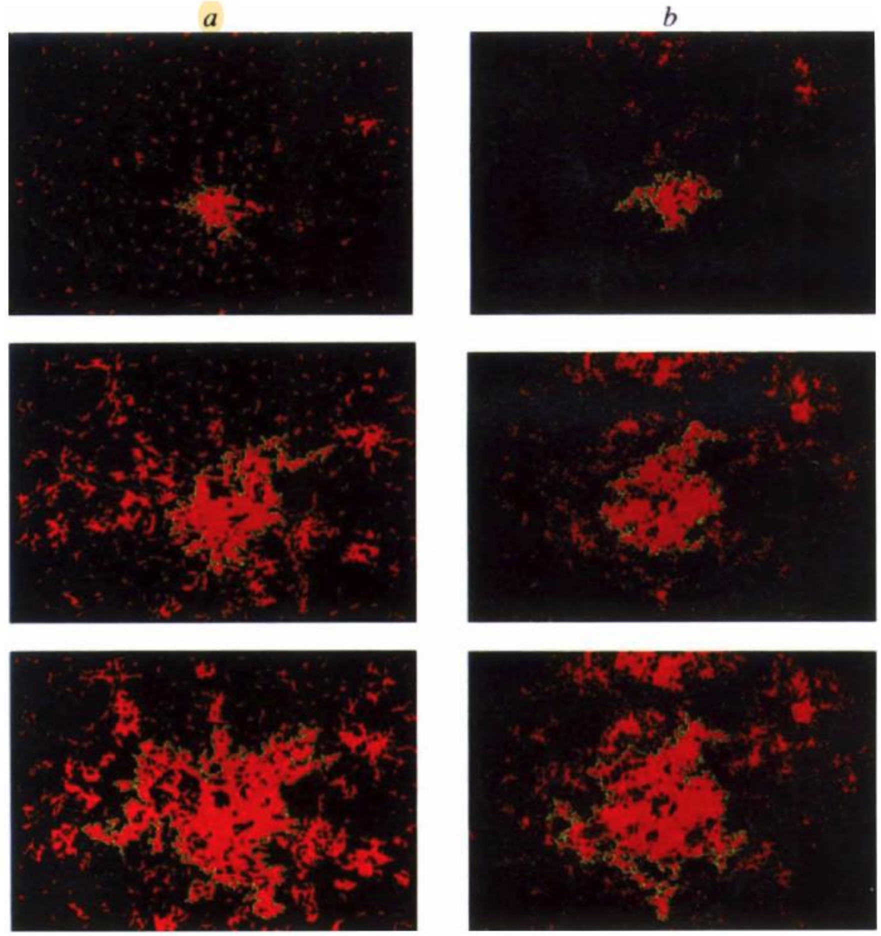
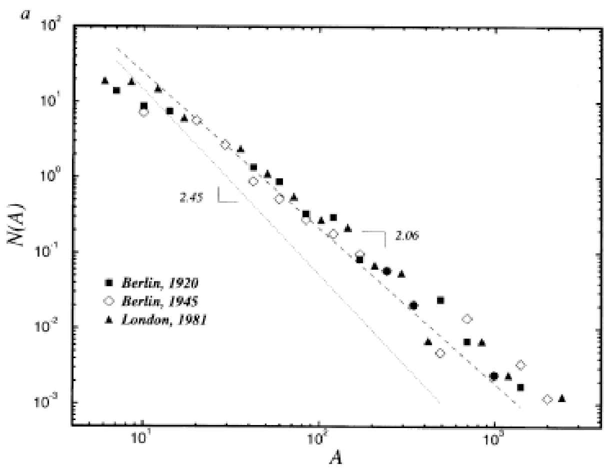
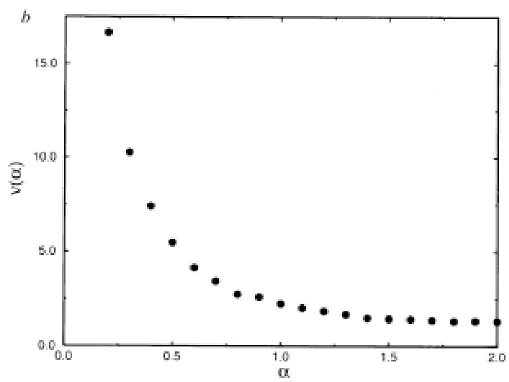



FIG. 2 Qualitative comparison between the actual urban data and the proposed model. *a*, Three steps of the growth with time of Berlin and surrounding towns. Data are shown for the years 1875, 1920 and 1945 (from top to bottom). *b*, Dynamical urban simulations of the proposed model. We fix the value of the correlation exponent to be $\alpha = 0.05$ (strongly correlated case), and choose the occupancy probability $p(r)$ to correspond to the density profiles shown in Fig. 4. We use the same seed for the random-number generator in all panels.

The above shows the prediction of growth – utilizing the **strongly correlated case** (where $a = .05$) – of Berlin. On the left ("a") we see the real-world qualitative data, on the right ("b") we see the models performance.



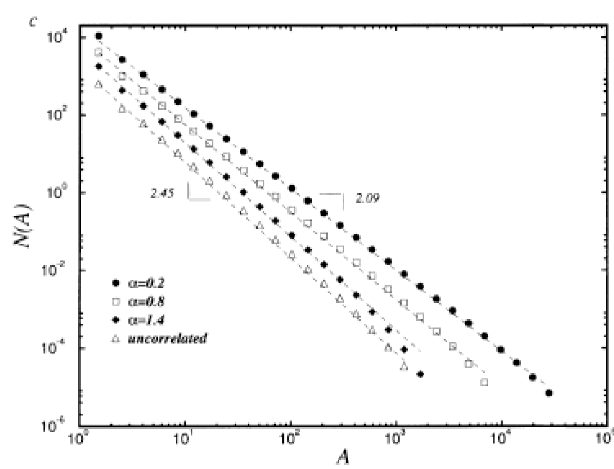




FIG. 3 *a*, Log–log plot of the area distribution $N(A)$ of the actual towns around Berlin and London. We first digitize the empirical data of Fig. 4.1 of ref. 13 (Berlin 1920 and 1945, shown in the last two panels of Fig. 2*a*), and Fig. 10.8 of ref. 1 (London 1981). Then, we count the number of towns that are covered by $A$ sites, putting the result in logarithmically spaced bins (of size $1.2^k$, with $k = 1, 2, \ldots, 16$), and averaging over the size of the bin. A power law is observed for the area distributions of both urban systems. The dotted line shows the predictions of our model for the uncorrelated case (slope, 2.45), while the dashed line gives results for the strongly correlated case (slope, 2.06). Note that the area distributions for both cities agree much better with the strongly correlated case ($\alpha \to 0$). *b*, Connectedness length exponent $\nu(\alpha)$ as a function of the correlation exponent $\alpha$. *c*, Log–log plot of the area distribution $N(A)$ calculated for the present model for different degrees of correlation. From top to bottom, $\alpha = 0.2$, $\alpha = 0.8$, $\alpha = 1.4$, and uncorrelated case. The linear fits correspond to the predictions of equation (1) using the values of $\nu(\alpha)$ from *b*, and $d_f = 1.9$.

These are additional plots which further breakdown that the power-law distribution maps well to their model as we see this function with respect to the cumulative distribution area $N(A)$ of area $A$ – which is defined as…

$$N(A) \equiv \int_0^{p_c} n(A,p)\,dp \sim A^{-(\tau + 1/d_f \nu)} \qquad (1)$$

Here, $n(A,p) \sim A^{-\tau} g(A/A_0)$ is defined to be the average number of clusters containing $A$ sites for a given $p$ at a fixed distance $r$, and $\tau = 1 + 2/d_f$. Here, $A_0(r) \sim |p(r) - p_c|^{-d_f \nu}$ corresponds to the maximum typical area occupied by a cluster situated at a distance $r$ from the CBD, and $g(A/A_0)$ is a scaling function that decays rapidly (exponentially) for $A > A_0$. The exponent $\nu = \nu(\alpha)$ is defined by $\xi(r) \sim |p(r) - p_c|^{-\nu}$, where $\xi(r)$ is the connectedness length that represents the mean linear size of a cluster at a distance $r > r_f$ from the CBD.

And finally, a plot representing the density of occupied urban areas for the three different stages in Berlin's growth. The description describes it better than I can, but it shows that the density gradient parameter scales as the size of the city increases.

FIG. 4 Semi-log plot of the density of occupied urban areas $\rho_A(r) = e^{-\lambda r}$ for the three different stages in the growth of Berlin shown in Fig. 2*a*. Least-squares fits yield the results $\lambda \approx 0.030$, $\lambda \approx 0.012$, and $\lambda \approx 0.009$, respectively, showing the decrease of $\lambda$ with time. We use these density profiles in the dynamical simulations of Fig. 2*b*. In the context of our model, this flattening pattern can be explained as follows. The model of percolation in a gradient can be related to a dynamical model of units (analogous to the development units in actual cities) diffusing from a central seed or core[9]. In this dynamical system, the units are allowed to diffuse on a two-dimensional lattice by hopping to nearest-neighbour positions. The density of units at the core remains constant: whenever a unit diffuses away from the core, it is replaced by a new unit. The density of units is analogous to $\rho_A(r)$, which in turn is proportional to[1] the population density $\rho(r)$. A diffusion front (defined as the boundary of the cluster of units that is linked to the central core) evolves with time. The diffusion front corresponds to the urban boundary of the central city. The static properties of the diffusion front of this system were found to be the same as those predicted by the gradient percolation model[9]. Moreover, the dynamical model can explain the decrease of $\lambda(t)$ with time that is observed empirically. As the diffusion front situated around $r_f$ moves away from the core, the city grows and the density gradient decreases because $\lambda(t) \propto 1/r_f$.